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What Can We Learn About Quark Binding From Perturbation Theory?

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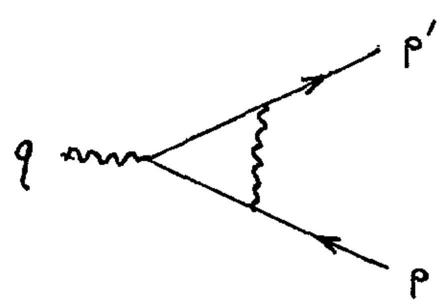
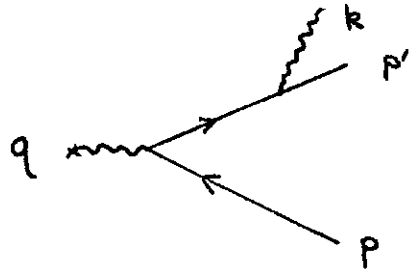
What can we learn about
quark binding from perturbation theory?

Perturbation theory certainly cannot be expected to provide us with any reliable detailed information about an inherently strong coupling problem such as quark binding. Nevertheless, we might reasonably ask whether or not perturbation theory provides us with any hints about an underlying quark binding mechanism when applied to the "standard" model of quarks interacting with non-Abelian vector gluons (Quantum Chromodynamics or Q. C. D.).

In an attempt to investigate this question, Tom Appelquist, Hanna Kluberg-Stern, Mike Roth and myself set out to examine the classic Block-Nordsieck program[1,2,3] in the framework of Q. C. D. Let me remind you that in Quantum Electrodynamics the Block-Nordsieck program assures us that the infrared divergences associated with virtual corrections are cancelled by corresponding divergences in the emission of undetected photons whose total energy is less than the energy resolution ΔE of the detector. For the vertex in second order Q. E. D. this means that when we form a partial cross section the infrared divergences of the virtual exchange diagram:



are cancelled by the divergence of the emission diagram

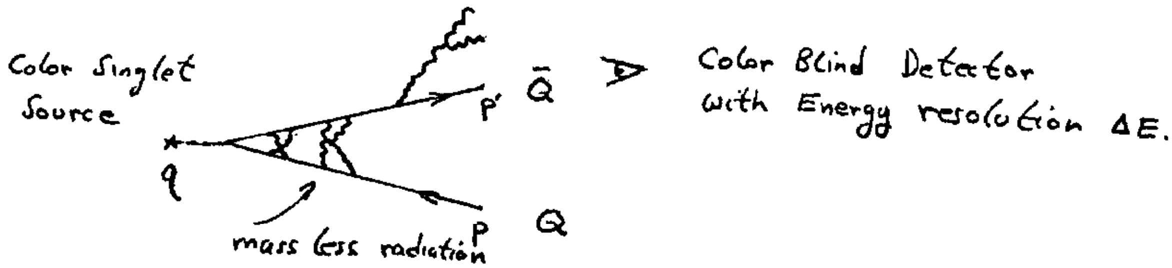


when the undetected emitted photon has energy less than the energy resolution of the detector. For $q^2 > (2m_e)^2$, where m_e is the electron mass, the fact that the partial cross section is free of infrared divergence implies that in Q. E. D. it is possible to produce an electron-positron pair and to detect either the electron or positron at some distant point provided we recognize the fact that the process involves undetectable radiation of soft photons.

In Q. C. D. we want to examine essentially the same process--production of a quark-antiquark pair by a color singlet current,

$$J_\mu(x) = \sum_i \bar{q}_i(x) \gamma_\mu q_i(x)$$

Instead of accompanying real and virtual photon radiation as we had in the Q. E. D. case, we now have gluon radiation in the Q. C. D. case. The detector has a finite energy resolution ΔE, and in order to obtain a gauge invariant cross section we assume the quark detector to trigger on some aspect of quark flavor. The quark detector therefore produces a sum over color states. Our experiment then looks like



where $q^2 > (2\mu)^2$ for μ = non-zero quark mass.

Some differences with Q. E. D. exist of course. The most important of these is that in Q. C. D. the massless fields couple to themselves. This means that the coupling must be normalized at some off-mass-shell momentum point $p^2 = -M^2$ so that the renormalized coupling is denoted by g(M). Another difference with Q. E. D. is the nature of the infrared cutoff. The device of using a photon mass as a cutoff cannot be extended to field theories with a non-Abelian gauge invariance. The best way to cutoff the theory and maintain gauge invariance is to dimensionally continue to $4 + \epsilon$ dimensions where ϵ is complex and is taken to zero at the end.

The object to be computed is a unitless transition probability appropriate to the experiment described earlier,

$$R_{\Delta E}\left(\frac{E}{M}, \frac{\Delta E}{M}, \frac{\mu}{M}, \epsilon, g(M)\right) \propto \int_{\Delta E} dE \left(\frac{d\sigma}{dE}\right)$$

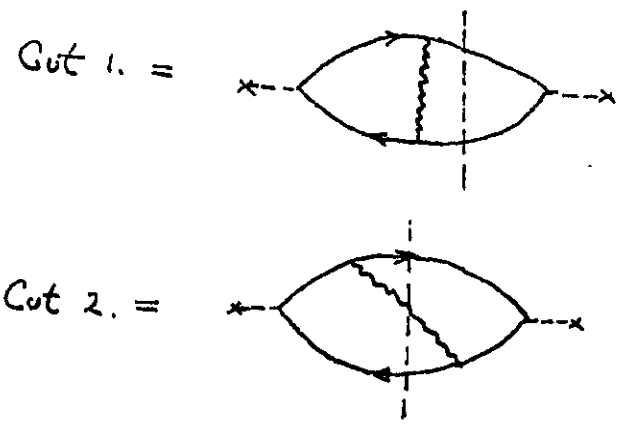
where $R_{\Delta E}$ is normalized by the Born amplitude and E represents the energies (center of mass, quark energy). The calculation is organized by grouping together the different unitarity cuts of each diagram contributing to

$$\Pi_{\mu\nu}(q) = \int (dx)\; e^{iq\cdot x} \langle 0|\, T(J_\mu(x)\, J_\nu(0))\,|0\rangle$$

The phase-space integral over the quark momentum is restricted by the detector kinematics which include a finite energy resolution. For simplicity all calculations are carried out in Feynman gauge. The final result is independent of gauge choice.

On the two-loop level $\pi_{\mu\nu}(q)$ contains only ordinary Q. E. D. diagrams so that $R_{\Delta E}$ is clearly finite. It is instructive to repeat the argument for finiteness in the simple case of the vertex correction contribution to $\pi_{\mu\nu}(q^2)$:

Altogether there are four unitarity cuts of this diagram. The following two:



plus two others which are reflections of 1 and 2. The rules for unitarity cuts are:

1. A progator factor of $i/p^2-m^2+i\epsilon$ is replaced by $2\pi\delta_+(p^2-m^2)$ $[2\pi\delta_-(p^2-m^2)]$ when the momentum runs from left to right

(right to left) across the cut.

2. By convention, every factor to the right of a cut is complex conjugated.

In order to see the cancellation of infrared divergences between cuts 1 and 2 it is necessary to perform only the (dk_0) integration for cut 1. This is done by complex integration in the complex k_0 plane. Closing in the lower complex k_0 plane produces two possible infrared divergent contributions, from the pole due to the quark propergator $1/k^2+2p\cdot k+i\epsilon$ we get,

$$[(2\pi)^2 C_N i g^2](-2\pi i) \oint \int \frac{(d\vec{K})}{(2\pi)^4} F(p,p') \, \mathrm{Tr}[\cdots] \times$$

$$\times \frac{1}{2\sqrt{①}} \frac{1}{(-p_0+\sqrt{①})^2-\vec{K}^2-2p_0(-p_0+\sqrt{①})-2\vec{p}\cdot\vec{K}} \frac{1}{(-p_0+\sqrt{①})^2-\vec{K}^2}$$

where I have combined phase-space integrations appropriate to the detector as

$$\oint F(p,p') \equiv \int \frac{(dp)}{(2\pi)^4} \int \frac{(dp')}{(2\pi)^4} \left(\delta(q-p+p') \, \delta_+(p^2-\mu^2) \, \delta_-(p'^2-\mu^2) \right|_{\text{Detector}}$$

and in the center of mass system,

$$p_0 = -p_0' = \sqrt{\vec{p}^{\,2}+\mu^2}$$

$$\sqrt{①} = \sqrt{(\vec{p}+\vec{K})^2+\mu^2}$$

The integration over $(d\vec{k})$ is divergent for small $\vec{k}$, however the divergent piece of the integral can be written as

$$\int (d\vec{k}) \; \frac{-1}{2p_0\left(\frac{\vec{p}\cdot\vec{k}}{p_0}\right)+2\vec{p}\cdot\vec{k}} \; \frac{1}{\left(\frac{\vec{p}\cdot\vec{k}}{p_0}\right)^2-\vec{k}^2}$$

which is odd in $\vec{k}\to-\vec{k}$ and therefore zero. The other infrared divergent contribution to cut 1 comes from the pole due to the $1/k^2+i\epsilon$ factor,

$$[(2\pi)^2 C_N\, i g^2]\,(-2\pi i) \oint \int \frac{(d\vec{k})}{(2\pi)^4}\; F(p,p')\; Tr\,[\cdots] \;\times$$

$$\times \frac{1}{2|\vec{k}|}\; \frac{1}{2p_0|\vec{k}|-2\vec{p}\cdot\vec{k}}\; \frac{1}{-2p_0|\vec{k}|-2\vec{p}\cdot\vec{k}} \;.$$

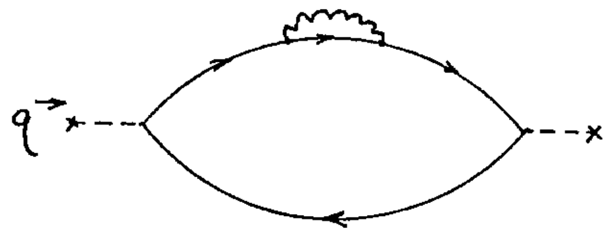
The integral of cut 2 can be evaluated directly as

$$-(2\pi)^3 C_N\, g^2 \oint \int \frac{(d\vec{k})}{(2\pi)^4}\; F(p, p'^2+2p'k)\; Tr\,[\cdots] \;\times$$

$$\times \frac{1}{2|\vec{k}|}\; \frac{1}{2p_0|\vec{k}|-2\vec{p}\cdot\vec{k}}\; \frac{1}{2(-|\vec{k}|-\sqrt{①})|\vec{k}|-2\vec{p}\cdot\vec{k}}$$

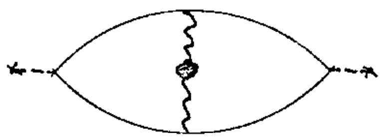
the divergent piece of this integral exactly cancels the remaining divergent integral of cut 1.

In addition to the vertex correction contribution to $\pi_{\mu\nu}(q^2)$ we also have a self-energy insertion correction on the two loop level,

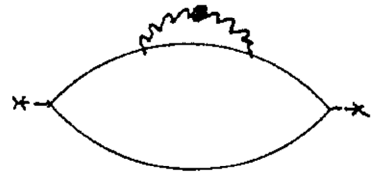


This diagram by itself is not infrared divergent, however when it is combined with the quark mass counter-term it becomes infrared divergent if it is subtracted at the quark mass-shell. Fortunately, our previous argument can be applied to the infrared divergent combination if we replace $1/\not{k}' - \mu + i\epsilon$ by $1/\not{k} - \mu + i\epsilon$ throughout the calculation. The result is the same: all infrared divergences cancel amoung the cuts of the diagram. Thus at the two loop level all infrared divergences cancel in an appropriately defined cross-section. Notice that if our detectors could distinguish color, we could not combine the various cuts as we did. We would be left with an infrared divergence.

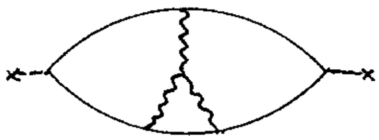
On the three-loop level there are an enormous number of diagrams to consider. It turns out that all but four are found in ordinary Q. E. D. and thus are known to have infrared divergences which cancel between the various cuts of each diagram.[4] The four new diagrams which are unique to Q. C. D. are



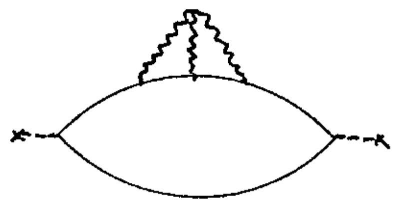
3-a



3-b



3-c



3-d

where

denotes the one-loop corrections to the gluon propagator. This object produces new infrared divergences not found in Q. E.D. because of the self-coupling of the massless gluon fields.

It turns out that by using the dispersion relation for the gluon propagator we can reduce the first two diagrams to the two-loop situation. Consider diagram 3-a, the complete gluon propagator may be written as

$$(-i)\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2+i\epsilon}\,d(k^2)$$

where $d(k^2)$ satisfies (see below)

$$\frac{d\left[\frac{k^2}{M^2},\epsilon,g(M)\right]}{k^2+i\epsilon}=\frac{d[0,\epsilon,g(M)]}{k^2+i\epsilon}+\int_0^\infty d\lambda^2\,\frac{\pi(\lambda^2)}{k^2-\lambda^2+i\epsilon}$$

The sum over unitarity cuts (with proper restrictions for the detector) of the first diagram is

$$\sum_{\Delta E \text{ cuts}} \pi_{\mu\nu}^{(3-a)}(q) = d^{(2-\text{loop})}[0,\epsilon,g(M)] \sum_{\Delta E \text{ cuts}} \pi_{\mu\nu}^{(2-\text{loop})}(q,0)$$
$$+ \int_0^\infty d\lambda^2\, \pi(\lambda^2) \sum_{\Delta E \text{ cuts}} \pi_{\mu\nu}^{(2-\text{loop})}(q,\lambda)\,.$$

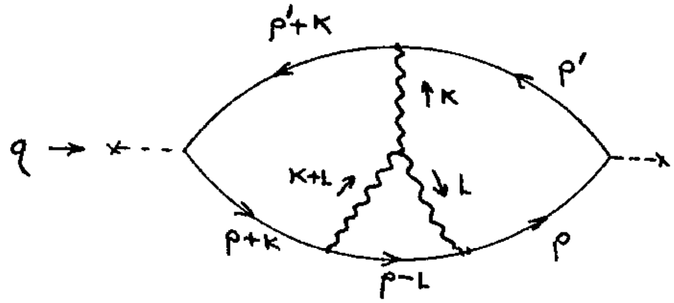
where $\pi_{\mu\nu}^{(2\text{-loop})}(q,\lambda)$ corresponds to the two-loop vertex correction diagram considered earlier except with a mass λ for the internal vector boson. The argument given earlier shows that the sum over cuts is finite even for $\lambda = 0$. Thus the above equation is free of infrared divergence as $\epsilon \to 0$ provided

$$d^{(2-\text{loop})}[0,\epsilon,g(M)] + \int_{\to 0} d\lambda^2\, \pi(\lambda^2)$$

is finite. This is indeed the case because $d[k^2/M^2, \epsilon, g(M)]$ is finite as $\epsilon \to 0$ for $k^2/M^2 \neq 0$ since it is normalized at $k^2 = -M^2$. This arguement has only one slight complication: the dispersion intergral cannot really be taken down to $\lambda = 0$ because the origin is an essential singularity. The same conclusion follows, however, if use is made of a small circle of radius δ about the origin in the complex λ^2 plane.

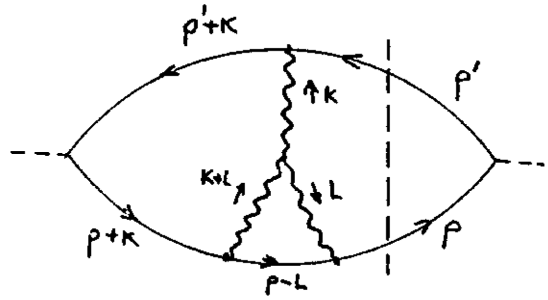
The next two diagrams, 3-c and 3-d are much more difficult to deal with. Just as in the two-loop case the sum of cuts of diagram 3-d can be proven to be free of divergences by essentially the same arguement that applies to 3-c. The same kind of analysis that I used in the two-loop case must be applied to 3-c with the added complication that there are now angular infrared divergences that are unique to theories of self-coupled massless gauge fields.

To begin, we want to consider the diagram of 3-c with rootings as follows:

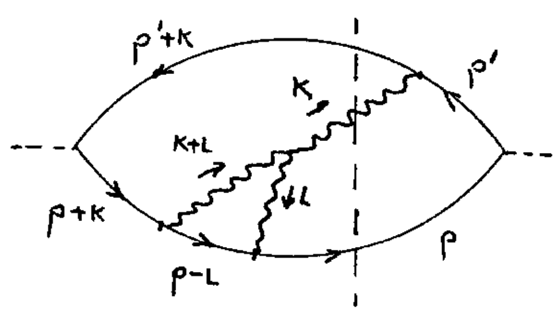


Altogether there are ten cuts of this diagram. The following five plus five more which are just reflections of the first five:

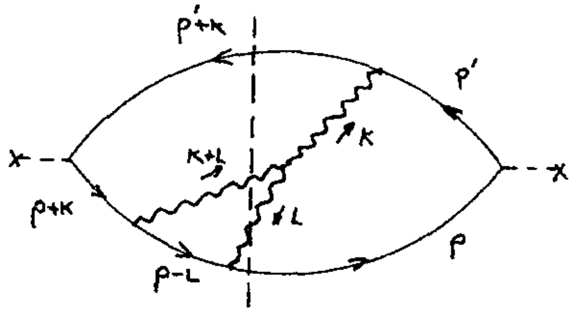
Cut 1.



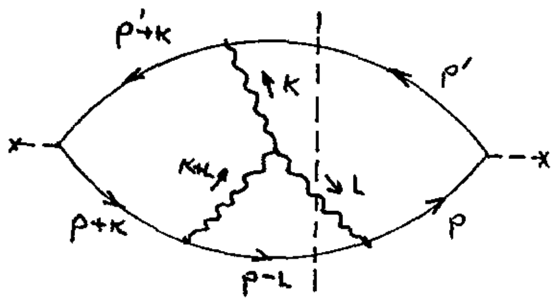
Cut 2.



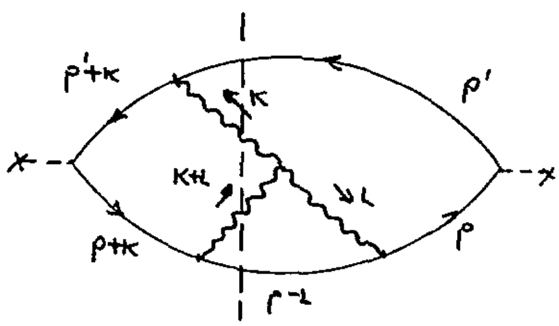
Cut 3.



Cut 4.



Cut 5.



The procedure is to perform the integrations over (dk_o) and $(d\ell_o)$ and then to isolate all remaining infrared divergences. The proof of infrared finiteness is to show that as $\epsilon \to 0$ the divergent parts of the remaining integrals cancel among these five different cuts.

The first cut is the most difficult because both integrations must be done as integrations in the complex plane. First let me make the notation as compact as possible:

Each diagram contains a trace factor of

$$\mathrm{Tr}[\cdots]^{\mu\nu\lambda} \equiv \mathrm{Tr}[\gamma_\sigma(\not{p}+\not{k}+\mu)\gamma^\mu(\not{p}-\not{\ell}+\mu)\gamma^\nu(\not{p}+\mu) \times$$
$$\times\, \gamma^\sigma(\not{p}'+\mu)\gamma^\lambda(\not{p}'+\not{k}+\mu)]$$

and a tri-vector vertex factor of

$$\Gamma^{\mu\nu\lambda} \equiv (2L+K)^\lambda g^{\mu\nu} + (K-L)^\mu g^{\nu\lambda} - (2K+L)^\nu g^{\mu\lambda}$$

in the combination

$$N = \mathrm{Tr}[\cdots]^{\mu\nu\lambda}\,\Gamma_{\mu\nu\lambda}$$

inside of a phase-space integral

$$g^4(\text{Group theory factor})\int\frac{(dp)}{(2\pi)^4}\int\frac{(dp')}{(2\pi)^4}\int\frac{dK^{n-1}}{(2\pi)^4}\int\frac{dL^{n-1}}{(2\pi)^4}\, N \times$$
$$\times\left(\delta(q-p-p')(2\pi)^2\delta_+(A^2-\mu^2)\,\delta_-(B^2-\mu^2)\right|_{\text{detector}}$$

$$\equiv \oint F(A,B).$$

where A and B are quark momentum vectors. By $n = 4 + \epsilon$ I mean that the integrals are regulated by dimensional continuation with (complex) $\epsilon \to 0$ at the end. By $\left(\cdots\right|_{\text{detector}}$ I mean that the detector restrictions must also be inserted - i.e., the total energy of any emitted massless quanta is less than ΔE. In addition there may be angular restrictions applied to the detected quark. Then the diagram of cut 1 is

$$C_{1.} = \oint F(p, p') \int_{K_0, \ell_0} \frac{1}{K^2+2p\cdot K+i\epsilon} \; \frac{1}{K^2+2p'\cdot K+i\epsilon} \; \frac{1}{\ell^2-2p\cdot\ell+i\epsilon} \times$$

$$\times \frac{1}{K^2+i\epsilon} \; \frac{1}{\ell^2+i\epsilon} \; \frac{1}{(K+\ell)^2+i\epsilon}$$

where $\int_{K_0} = \int_{-\infty}^{+\infty} \frac{dK_0}{2\pi}$ etc. If I want to think of doing the $d\ell_0$ integration first I may write this as a sum of three terms

$$C_{1.} = C_{1.}^{1.} + C_{1.}^{2.} + C_{1.}^{3.} = \oint F(p, p') \int_{K_0, \ell_0} (-2\pi i) \left\{\cdots\right\} \times$$

$$\times \frac{1}{K^2+i\epsilon} \; \frac{1}{K^2+2p\cdot K+i\epsilon} \; \frac{1}{K^2+2p'\cdot K+i\epsilon}$$

where (in the complex ℓ_0 plane)

$$\{\cdots\} = \delta_+^{\ell}(\ell^2 - 2p\cdot\ell + i\epsilon)\frac{1}{2p\cdot\ell}\,\frac{1}{K^2+2K\cdot\ell+2p\cdot\ell}$$
$$+\,\delta_+^{\ell}(\ell^2+i\epsilon)\,\frac{-1}{2p\cdot\ell}\,\frac{1}{K^2+2K\cdot\ell}$$
$$+\,\delta_+^{\ell}((K+\ell)^2+i\epsilon)\,\frac{-1}{K^2+2K\cdot\ell+2p\cdot\ell}\,\frac{-1}{K^2+2K\cdot\ell}$$

The delta functions force ℓ_o to have the values (in c.m. frame)

$$\ell_o = p_o + \sqrt{①} - i\epsilon/2\sqrt{①} \qquad \text{in} \quad C_{1.}^{1.}$$

$$\ell_o = |\vec{\ell}| - i\epsilon/2|\vec{\ell}| \qquad \text{in} \quad C_{1.}^{2.}$$

$$\ell_o = -K_o + |\vec{K}+\vec{\ell}| - i\epsilon/2|\vec{K}+\vec{\ell}| \quad \text{in} \quad C_{1.}^{3.}$$

$$\sqrt{①} = \sqrt{(\vec{p}-\vec{\ell})^2+\mu^2}.$$

I must be careful to retain the $i\epsilon$ terms as $\epsilon\to 0^+$ because this generates restrictions on which poles I pick up when I do the dk_o integration.

Next I do the dk_o integration and let $\epsilon\to 0^+$ in the denominators. For $C_{1.}^{1.}$ there are four poles in the complex k_o plane which I label by means of a second index:

$$C_{1.}^{1.} = C_{1.}^{1.,1.} + C_{1.}^{1.,2.} + C_{1.}^{1.,3.} + C_{1.}^{1.,4.}$$

The first is at $k_o = |\vec{k}+\vec{\ell}| - \sqrt{①} - p_o$ and has no infrared singularities of any kind. The second is at $k_o = |\vec{k}|$ and can be written as

$$(-2\pi i)^2 \oint F(p,p') \; [\cdots]$$

where [---] is obtained by substitution of $\ell_o = p_o + \sqrt{①}$ and $k_o = |\vec{k}|$ in the above expressions. Inspection shows that this term has a $D = 0$ infrared divergence in the $d\vec{k}$ integration as $|\vec{k}| \to 0$ for $|\vec{\ell}| \not\to 0$ and no other divergence. I shall label this source of divergence $C^{1.,2.}_{1.}(k_s\, \ell_h, 0)$ where the subscript stands for cut 1., and the two superscripts denote the particular term. The two variables describe the limit which generates the infrared divergence, and the zero means that the divergence is logarithmic. Clearly a single term may generate several infrared divergences each from different regions of the integration variables. To denote this I will express the contribution of a given term as sum of terms each one of which has an infrared divergence arising from a particular limit of the integration variables. Finally I denote the degree of divergence by the last variable. The term described above as $C^{1.,2.}_{1.}(k_s\, \ell_h, 0)$ will eventually be canceled by a term in cut 2 referred to as $C^{1.}_{2.}(k_s\, \ell_h, 0)$ for which $k_o = |\vec{k}|$ and $\ell_o = p_o + \sqrt{①}$.

The third pole in the k_o plane is at

$$K_o = -p_o + \sqrt{②} \qquad \text{where}$$

$$\sqrt{②} = \sqrt{(\vec{p}+\vec{k})^2 + \mu^2}$$

Remember all $C_{1.}^{1.}$ terms have $\ell_o = p_o + \sqrt{①}$. This term has a $D=0$ divergence for $k_s \ell_h$ but does not contribute in this limit because it is odd under $\vec{k} \to -\vec{k}$. Thus $C_{1.}^{1.,3.}(k_s \ell_h, 0) = 0$. The term $C_{1.}^{1.,4.}$ is at $k_o = p_o + \sqrt{②}$ which is bounded away from the origin in the k_o plane; thus there is no infrared divergence.

For the term $C_{1.}^{2.}$ ($\ell_o = |\vec{\ell}|$) we begin to encounter a more complex situation. There are three poles in the complex k_o plane which produce infrared divergence. The first one of these, $k_o = |\vec{k} + \vec{\ell}| - |\vec{\ell}|$, produces a very complicated term

$$C_{1.}^{2.,1.} = C_{1.}^{2.,1.}(k_s \ell_h, -1) + C_{1.}^{2.,1.}(k_s \ell_s, 0)$$

$$+ C_{1.}^{2.,1.}(\vec{k} \parallel \vec{\ell}, 0)$$

where $C_{1.}^{2.,1.}(\vec{k} \parallel \vec{\ell}, 0)$ denotes an angular logarithmic divergence which occurs for $\vec{k}$ parallel to $\vec{\ell}$. Fortunately the entire term, $C_{1.}^{2.,1.}$, is identically canceled by the term $C_{1.}^{3.,3.}$ which has the same values of $\ell_o = |\vec{\ell}|$ and $k_o = |\vec{k} + \vec{\ell}| - |\vec{\ell}|$. At $k_o = |\vec{k}|$ we encounter a term, $C_{1.}^{2.,2.}$, which contains the factor

$$\frac{1}{2|\vec{k}||\vec{\ell}| - 2\vec{k}\cdot\vec{\ell}}$$

which generates an angular logarithmic divergence when $\vec{k}$ is parallel to $\vec{\ell}$ (essentially the same term generates $C_{1.}^{2.,1.}(\vec{k} \parallel \vec{\ell}, 0)$). Altogether we have the following infrared divergence structure:

$$C_{1.}^{2.,2.} = C_{1.}^{2.,2.}(k_s\ell_h, -1) + C_{1.}^{2.,2.}(k_h\ell_s, 0)$$

$$+ C_{1.}^{2.,2.}(k_s\ell_s, 0) + C_{1.}^{2.,2.}(\vec{k} \parallel \vec{\ell}, 0).$$

The piece $C_{1.}^{2.2.}(k_s\ell_h, -1)$ combines with $C_{1.}^{3.,4.}(k_s\ell_h, -1)$ (both have $k_o = |\vec{k}|$ and $\ell_o = |\vec{\ell}|$ in this limit) to produce a remainder, $\left[C_{1.}^{2.,2.}(k_s\ell_h, -1) + C_{1.}^{3.,4.}(k_s\ell_h, -1)\right]$ which has only a $D = 0$ divergence. The remainder finally cancels with a similar combination of terms in cut 2., $\left[C_{2.}^{2.}(k_s\ell_h, -1) + C_{2.}^{3.}(k_s\ell_h, -1)\right]$. The term $C_{1.}^{2.,2.}(k_h\ell_s, 0)$ cancels with $C_{1.}^{3.,2.}(k_h\ell_s, 0)$ when $\vec{\ell} \to -\vec{\ell}$ in the $k_h\ell_s$ limit. The "double soft" term $C_{1.}^{2.2.}(k_s\ell_s, 0)$ cancels against $C_{2.}^{2.}(k_s\ell_s, 0)$, and the angular piece, $C_{1.}^{2.2.}(\vec{k} \parallel \vec{\ell}, 0)$ will be discussed later.

The last contributing pole in the k_o plane is at $k_o = -p_o + \sqrt{②}$ and produces $C_{1.}^{2.,3.} = C_{1.}^{2.,3.}(k_s\ell_h\ -1) + C_{1.}^{2.,3.}(k_s\ell_s, 0)$. The term $C_{1.}^{2.,3.}(k_s\ell_h, -1)$ combines with $C_{1.}^{3.,5.}(k_s\ell_h, -1)$ to produce a $D = 0$ remainder which is explicitly odd under $\vec{k} \to -\vec{k}$ and hence zero. The term $C_{1.}^{2.,3.}(k_s\ell_s, 0)$, in which $|\vec{k}|$ and $|\vec{\ell}|$ both go to zero, can be seen to cancel $C_{1.}^{3.,5.}(k_s\ell_s, 0)$ if we note that the transformation,

$$\vec{k} \to -\vec{k}'$$
$$\vec{\ell} \to \vec{\ell}' + \vec{k}'$$

leaves the numerator factor, N, invariant <u>in this limit only</u>.

Next we come to the terms of $C^{3.}_{1.}$ ($\ell_o = -k_o + |\vec{k} + \vec{\ell}|$) several of which I have already mentioned. In this case, there are five poles in the complex k_o plane which generate infrared divergences. Briefly we have:

1.) $C^{3.,1.}_{1.}$ at $k_o = |\vec{k} + \vec{\ell}| - p_o + \sqrt{①}$; $\ell_o = p_o - \sqrt{①}$ with

$$C^{3.,1.}_{1.} = C^{3.,1.}_{1.}(k_h \ell_s,\ 0) + C^{3.,1.}_{1.}(k_s \ell_s,\ 0).$$

The term $C^{3.,1.}_{1.}(k_h \ell_s,\ 0)$ is odd under $\vec{\ell} \to -\vec{\ell}$ and vanishes in the $k_h \ell_s$ limit. The term $C^{3.,1.}_{1.}(k_s \ell_s,\ 0)$ cancels with a term of cut 4., $C^{1.}_{4.}(k_s \ell_s,\ 0)$, if we make the transformation

$$\vec{k} \to -\vec{k}'$$
$$\vec{\ell} \to \vec{\ell}' + \vec{k}'$$

in the term of cut 4.

2.) $C^{3.,2.}_{1.}$ at $k_o = |\vec{k} + \vec{\ell}| + |\vec{\ell}|$; $\ell_o = -|\vec{\ell}|$ with

$$C^{3.,2.}_{1.} = C^{3.,2.}_{1.}(k_h \ell_s,\ 0) + C^{3.,2.}_{1.}(k_s \ell_s,\ 0)$$

$$+ C^{3.,2.}_{1.}(\vec{k} \parallel \vec{\ell},\ 0).$$

The term $C^{3.,2.}_{1.}(k\ \ \ell_s,\ 0)$ cancels $C^{2.,2.}_{1.}(k_h \ell_s,\ 0)$ as already mentioned. The term $C^{3.,2.}_{1.}(k_s \ell_s,\ 0)$ cancels with the only term of cut 3. in the $k_s \ell_s$ limit, $C_{3.}(k_s \ell_s,\ 0)$. The angular divergence I leave for later.

3.) $C_{1.}^{3.,3.}$ at $k_o = |\vec{k} + \vec{\ell}| - |\vec{\ell}|$; $\ell_o = |\ell|$ cancels $C_{1.}^{2.,1.}$ as promised.

4.) $C_{1.}^{3.,4.}$ at $k_o = |\vec{k}|$, $\ell_o = -|\vec{k}| + |\vec{k} + \vec{\ell}|$ with

$$C_{1.}^{3.,4.} = C_{1.}^{3.,4.}(k_s \ell_h, -1) + C_{1.}^{3.,4.}(k_s \ell_s, 0)$$
$$+ C_{1.}^{3.,4.}(\vec{k} \parallel \vec{\ell}, 0).$$

The term $C_{1.}^{3.,4.}(k_s \ell_h, -1)$ combines with $C_{1.}^{2.,2.}(k_s \ell_h, -1)$ as already discussed. The "double-soft" term $C_{1.}^{3.,4.}(k_s \ell_s, 0)$ cancels a term of cut 2., $C_{2.}^{3.}(k_s \ell_s, 0)$.

5.) $C_{1.}^{3.,5.}$ at $k_o = -p_o + \sqrt{②}$; $\ell_o = -p_o + \sqrt{②} + |\vec{k} + \vec{\ell}|$ with

$$C_{1.}^{3.,5.} = C_{1.}^{3.,5.}(k_s \ell_h, -1) + C_{1.}^{3.,5.}(k_s \ell_s, 0).$$

The $C_{1.}^{3.,5.}(k_s \ell_h, -1)$ term combines with $C_{1.}^{2.,3.}(k_s \ell_h, -1)$, and the $C_{1.}^{3.,5.}(k_s \ell_s, 0)$ term cancels $C_{1.}^{2.,3.}(k_s \ell_s, 0)$ (after transformation) as promised.

Our analysis of cut 1. is now complete, aside from angular divergences, and we turn to cut 2. which is much less complicated due to the presence of $\delta_+(k^2)$. Only three terms are present:

1.) $C_{2.}^{1.}$ at $k_o = |\vec{k}|$, $\ell_o = p_o + \sqrt{①}$ with

$$C_{2.}^{1.} = C_{2.}^{1.}(k_s \ell_h, 0).$$

This term cancels $C_{1.}^{1.,2.}(k_s \ell_h, 0)$.

2.) $C_{2.}^{2.}$ at $k_o = |\vec{k}|$, $\ell_o = |\vec{\ell}|$ with

$$C_{2.}^{2.} = C_{2.}^{2.}(k_s\ell_h, -1) + C_{2.}^{2.}(k_h\ell_s, 0)$$
$$+ C_{2.}^{2.}(k_s\ell_s, 0) + C_{2.}^{2.}(\vec{k} \parallel \vec{\ell}, 0).$$

The term $C_{2.}^{2.}(k_s\ell_h, -1)$ combines with $C_{2.}^{3.}(k_s\ell_h, -1)$ to become only $D = 0$ in this limit. The combination cancels $\left[C_{1.}^{2.,2.}(k_s\ell_h, -1) + C_{1.}^{3.,4.}(k_s\ell_h, -1)\right]$ as already explained. The term $C_{2.}^{2.}(k_h\ell_s, 0)$ is canceled by the single term of cut 3. in this limit, $C_{3.}(k_h\ell_s, 0)$. The double soft term $C_{2.}^{2.}(k_s\ell_s, 0)$ cancels $C_{1.}^{2.,2.}(k_s\ell_s, 0)$.

3.) $C_{2.}^{3.}$ at $k_o = |\vec{k}|$, $\ell_o = |\vec{k} + \vec{\ell}| - |\vec{k}|$ with

$$C_{2.}^{3.} = C_{2.}^{3.}(k_s\ell_h, -1) + C_{2.}^{3.}(k_s\ell_s, 0)$$
$$+ C_{2.}^{3.}(\vec{k} \parallel \vec{\ell}, 0).$$

As just mentioned, $C_{2.}^{3.}(k_s\ell_h, -1)$ combines with $C_{2.}^{2.}(k_s\ell_h, -1)$, and $C_{2.}^{3.}(k_s\ell_s, 0)$ cancels $C_{1.}^{3.,4.}(k_s\ell_s, 0)$.

Cut 3. contains only one term because of the two delta functions. It is $C_{3.}$ at $k_o = |\vec{\ell}| + |\vec{k} + \vec{\ell}|$, $\ell_o = -|\vec{\ell}|$ with

$$C_{3.} = C_{3.}(k_h\ell_s, 0) + C_{3.}(k_s\ell_s, 0)$$
$$+ C_{3.}(\vec{k} \parallel \vec{\ell}; 0).$$

The term $C_{3.}(k_h\ell_s, 0)$ cancels $C_{2.}^{2.}(k_h\ell_s, 0)$ after we let $\vec{\ell} \to -\vec{\ell}$, and the term $C_{3.}(k_s\ell_s, 0)$ cancels $C_{1.}^{3.,2.}(k_s\ell_s, 0)$.

Cut 4. has a $\delta_+(\ell^2)$ so that $\ell_o = |\vec{\ell}|$ and there are three terms:

1.) $C_{4.}^{1.}$ at $k_o = -p_o + \sqrt{②}$ with $C_{4.}^{1.} = C_{4.}^{1.}(k_s \ell_s, 0)$ as the only divergence. This term cancels $C_{1.}^{3.,1.}(k_s \ell_s, 0)$ after transformation

2.) $C_{4.}^{2.}$ at $k_o = |\vec{k} + \vec{\ell}| - |\vec{\ell}|$ with

$$C_{4.}^{2.} = C_{4.}^{2.}(k_h \ell_s, 0) + C_{4.}^{2.}((k + \ell)_s (k - \ell)_h, 0)$$
$$+ C_{4.}^{2.}(k_s \ell_s, 0) + C_{4.}^{2.}(\vec{k} \parallel \vec{\ell}, 0).$$

The term $C_{4.}^{2.}(k_h \ell_s, 0)$ cancels with the next term, $C_{4.}^{3.}(k_h \ell_s, 0)$, and $C_{4.}^{2.}((k + \ell)_s (k - \ell)_h, 0)$ cancels with the same limit of the single term of cut 5., $C_{5.}((k + \ell)_s (k - \ell)_h, 0)$. The term $C_{4.}^{2.}(k_s \ell_s, 0)$ changes sign under

$$\vec{K} \rightarrow -\vec{K}'$$
$$\vec{\ell} \rightarrow \vec{K}' + \vec{\ell}'$$

in the $k_s \ell_s$ limit and is therefore zero.

3.) $C_{4.}^{3.}$ at $k_o = |\vec{k}|$ with

$$C_{4.}^{3.} = C_{4.}^{3.}(k_s \ell_h, 0) + C_{4.}^{3.}(k_h \ell_s, 0)$$
$$+ C_{4.}^{3.}(k_s \ell_s, 0) + C_{4.}^{3.}(\vec{k} \parallel \vec{\ell}, 0).$$

The term $C_{4.}^{3.}(k_s \ell_h, 0)$ cancels the single term of cut 5. in the limit, $C_{5.}(k_s \ell_h, 0)$, if we let $\vec{k} \rightarrow -\vec{k}$. As just mentioned, $C_{4.}^{3.}(k_h \ell_s, 0)$ cancels with $C_{4.}^{2.}(k_h \ell_s, 0)$, and $C_{4.}^{3.}(k_h \ell_s, 0)$ is canceled by $C_{5.}(k_s \ell_s, 0)$ if we make the transformation

$$\vec{K} \rightarrow -\vec{K}'$$
$$\vec{\ell} \rightarrow \vec{K}' + \vec{\ell}'$$

Lastly, we have cut 5. which has only one term because of the two delta functions. These imply that

$$K_0 = -|\vec{K}|$$
$$\ell_0 = |\vec{K}| + |\vec{K} + \vec{\ell}|$$

and, we find

$$C_{5.} = C_{5.}(k_s \ell_h, 0) + C_{5.}((k+\ell)_s (k-\ell)_h, 0)$$
$$+ C_{5.}(k_s \ell_s, 0) + C_{5.}(\vec{k} \parallel \vec{\ell}, 0).$$

As already explained I find that $C_{5.}(k_s \ell_h, 0)$ cancels $C_{4.}^{3.}(k_s \ell_h, 0)$; $C_{5.}((k+\ell)_s(k-\ell)_h, 0)$ cancels $C_{4.}^{2.}((k+\ell)_s(k-\ell)_h, 0)$, and $C_{5.}(k_s \ell_s, 0)$ cancels $C_{4.}^{3.}(k_h \ell_s, 0)$.

There now remains only the angular divergences which are easy to treat. Define,

$$\cos\theta = \frac{\vec{K}\cdot\vec{\ell}}{|\vec{K}|\,|\vec{\ell}|}$$

for $|\vec{k}| \neq 0$ and $|\vec{\ell}| \neq 0$, and I consider cut 1. in detail first. There are three terms with angular divergence: $C_{1.}^{2.,2.}(\vec{k} \parallel \vec{\ell}, 0)$, $C_{1.}^{3.,2.}(\vec{k} \parallel \vec{\ell}, 0)$ and $C_{1.}^{3.,4.}(\vec{k} \parallel \vec{\ell}, 0)$. Each of these terms contains logarithmic divergence in the angular integration over $\cos\theta$. It is not difficult to check that $C_{1.}^{2.,2.}(\vec{k} \parallel \vec{\ell}, 0)$ has its divergence only at $\cos\theta = 1$ for any value of

$|\vec{k}|$ and $|\vec{\ell}|$ not zero. The term $C_{1.}^{3.,2.}(\vec{k} \parallel \vec{\ell}, 0)$ has divergence only when $\cos\theta = -1$ and $|\vec{k}| > |\vec{\ell}| > 0$, and $C_{1.}^{3.,4.}(\vec{k} \parallel \vec{\ell}, 0)$ has divergence at $\cos\theta = 1$ for $|\vec{k}|$, $|\vec{\ell}| > 0$ and at $\cos\theta = -1$ for $|\vec{k}| > |\vec{\ell}| > 0$. To denote these situations I write,

$$C_{1.}^{2.,2.}(\vec{k} \parallel \vec{\ell}, 0) = C_{1.}^{2.,2.}(1)$$

$$C_{1.}^{3.,2.}(\vec{k} \parallel \vec{\ell}, 0) = C_{1.}^{3.,2.}(-1, |\vec{k}| > |\vec{\ell}|)$$

$$C_{1.}^{3.,4.}(\vec{k} \parallel \vec{\ell}, 0) = C_{1.}^{3.,4.}(1) + C_{1.}^{3.,4.}(-1, |\vec{k}| > |\vec{\ell}|).$$

It is easy to show that $C_{1.}^{2.,2.}(1)$ cancels $C_{1.}^{3.,4.}(1)$ and $C_{1.}^{3.,2.}(-1, |\vec{k}| > |\vec{\ell}|)$ cancels $C_{1.}^{3.,4.}(-1, |\vec{k}| > |\vec{\ell}|)$. Almost precisely the same analysis applies to the angular divergences of cuts 2. and 3. and to the divergences of cuts 4. and 5.

What can we learn from this long analysis? At least two things are clear:

1.) It is essential that the infrared divergences be no stronger than they are in 4 dimensions. I expect the above cancellation of infrared divergences to break down in two or three dimensions.

2.) Because I had to use explicit properties of the numerator structure to show the cancellation, I assume that the cancellation may be interpreted as a direct result of the gauge invariance of the couplings.

Furthermore, if we are willing to assume that the cancellation continues to hold to all orders, we obtain the result that perturbation theory implies that it is indeed possible to separate quark flavor--hence no confinement.

Discussion

Question: (Dr. Stump)

Is it not true that the cancellation of infrared divergences you just described is, in some sense, obvious?

Answer:

In theories such as Q. E. D., the cancellation of infrared divergences for situations such as I have just described is often attributed to a well-known theorem of T. D. Lee and M. Nauenberg[5] which demonstrates that under very general conditions the cancellation of infrared divergences is a consequence of an elementary theorem of quantum mechanics which is independent of the explicit form of the Hamiltonian. Unfortunately for the case of Q. C. D. no one has yet been able to show that the theorem is applicable. The problem is that the theorem requires the existence of a cutoff parameter, μ, for which the theory is:

1.) Infrared divergent only for $\mu \rightarrow 0$.

2.) Well defined for $\mu \searrow 0$ in the sense that it possess a unitary S-matrix.

The device of inserting a gluon or photon mass works in Q. E. D. but is known not to work in the Yang-Mills case because of the non-hermitian nature of the ghost coupling. One might think of first going to Coulomb gauge and then introducing a gluon mass term, but this also does not work. The problem here is that the Coulomb gauge

Hamiltonian contains an infinite number of terms which serve to cancel the non-covariant vertices of the theory in the calculation of gauge invariants. If the infrared divergence of the theory is regulated by introduction of a gluon mass, the gauge invariance is destroyed, and the cancellation of the non-covariant vertices fails. This means that the theory is no longer well-defined because it is not equivalent to a truly renormalizable theory.

Question: (Dr. Yan)

Have you computed what $R_{\Delta E}$ actually is?

Answer:

No, because I have no way of finding how to extract the ΔE dependence <u>to all orders</u>. In Q. E. D. I know how to do this because Q. E. D. is "infrared free." Yang-Mills theory is not "infrared free" but is very complicated in this limit.

Question: (F. Wilczek)

Do you know if your use of Feynman gauge ensures that the infrared divergences in higher orders continue to cancel among the cuts of each diagram of $\pi_{\mu\nu}(q)$, or is there a cancellation of divergences between cuts of different diagrams of $\pi_{\mu\nu}(q)$?

Answer:

No, I don't know what happens in higher orders.

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